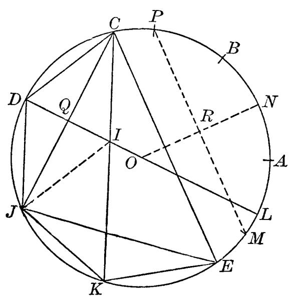
CONCERNING THE REGULAR INSCRIBED HEPTAGON.

By S. A. JOFFE, New York City.

Mr. J. Q. McNatt's article " A Geometrical Discussion of the Regular Inscribed Heptagon," which appeared in the January number of this MONTHLY, pages 13 and 14, contains a very interesting and ingenious method of arriving at what the author calls " The Heptagon Cubic," from the solution of which he shows that the square of the side of the regular heptagon equals $\frac{3}{4}$, as a very close approximation.

Without detracting from the value of the method, I should like to point out that some of the details in the process may be omitted or simplified, and it has occurred to me that it may be worth while to present here the method in an abbreviated form, thus making the matter more widely known. In preparing the following lines, I have endeavored to preserve as much of the original figure and of the author's text as was consistent with the object of simplification.

To calculate the length of the side of a regular inscribed heptagon in terms of the radius as a unit, suppose, in the accompanying figure, that the circumference is divided into seven equal parts at the points A, B, C, D, J, K, and E.



Draw the diameter DL and the chords JE, CJ, CK and CE, and let CJ and CK cut DL at Q and I respectively. Join I and J.

Let the length of the side of the regular inscribed heptagon be h units. We have $DL : DC :: DC : DQ$; hence $DQ = \frac{1}{2}h^2$. Moreover, $QC = \sqrt{DC^2 - DQ^2} = \sqrt{h^2 - \frac{1}{4}h^4} = \frac{1}{2}h\sqrt{4 - h^2}$. From the isosceles triangle DCI we have $CI = CD = h$, and $DI = 2DQ = h^2$; from the isosceles triangle CIJ we have $IJ = CI = h$, and $CJ = 2CQ = h\sqrt{4 - h^2}$, so that

$$JE = h\sqrt{4 - h^2}. \tag{1}$$

Now, by geometry $IL \times DI = CI \times IK$, and since $IL = DL - DI = 2 - h^2$, we have $(2 - h^2) \times h^2 = h \times IK$; hence

$$IK = 2h - h^3, \tag{2}$$

and since $CK = CI + IK = 3h - h^3$, and $CE = CK$, therefore

$$CE = 3h - h^3. \tag{3}$$

The two isosceles triangles CJE and IJK, having equal angles at E and K respectively, are similar. Hence we have $JE : CE :: JK : IK$, or, using (1), (2) and (3)

$$h\sqrt{4 - h^2} : (3h - h^3) :: h : (2h - h^3).$$

Hence

$$(2 - h^2)\sqrt{4 - h^2} = 3 - h^2. \tag{4'}$$

Squaring, we get

$$(4 - 4h^2 + h^4)(4 - h^2) = 9 - 6h^2 + h^4,$$

and expanding,

$$16 - 20h^2 + 8h^4 - h^6 = 9 - 6h^2 + h^4.$$

Finally, transposing and simplifying, we obtain the author's *Heptagon Cubic:*

$$7 - 14h^2 + 7h^4 - h^6 = 0. \tag{5}$$

Solving by Horner's method, we find $h^2 = .7530203962821\ldots = \frac{3}{4}$, approximately.

Remark. It will thus be seen that while there is introduced a new line IJ, we dispense with the consideration of the line OC, and with both the consideration and the computation of the author's lines SE, SJ, SK and SC. As a result, the equation (4′) appears in a much simpler form than the author's equation (4).

The approximate construction of the heptagon may also be simplified as follows:

Let M, N and P be three consecutive vertices of an inscribed regular hexagon. Draw the chord MP and the radius ON, and let MP meet ON in R. Then MR is, approximately, the length h of the side of the regular inscribed heptagon. The reason is self-evident: approximately, $h = \frac{1}{2}\sqrt{3}$, and MP, as the side of a regular inscribed triangle, $= \sqrt{3}$, so that $MR = \frac{1}{2}\sqrt{3}$, and therefore $MR = h$, approximately.

A PROBLEM IN NUMBER THEORY.

By GEO. A. OSBORNE, Massachusetts Institute of Technology.

§ 1. When is the sum of the squares of two successive integers a perfect square? The following are examples:

$$3^2 + 4^2 = 5^2, \quad 20^2 + 21^2 = 29^2. \quad \text{The next is } 119^2 + 120^2 = 169^2.$$

The numbers 3, 20, 119, . . . are the terms of a series

$$0,\ 3,\ 20,\ 119,\ 696,\ \cdots\ u_n,\ u_{n+1}, \tag{1}$$

where

$$u_{n+1} = 6u_n - u_{n-1} + 2. \tag{2}$$

This may be proved as follows: